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INTRODUCTION

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Contents

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Breaking the Fundamentals of Geometry. On axiomatic geometry, non-Euclidean geometries and non-Desarguesian planes

by Alèxia Escudero Ribó

Abstract

When we learn geometry, there are many concepts that we have not been taught and which we take for granted. How many people have considered if there really exists a unique parallel to a straight line through a point? Surely not everyone. But the fact is that this is not always true. This work focuses on alternative geometries to those we are used to: the so-called non-Euclidean geometries. This report summarizes my recent school research project, where we analyse the most important properties of these geometries, based on well-known books, and we deduce their axioms. In the main part of our research, we expose and understand several non-Desarguesian planes and geometries, i.e. planes that do not satisfy the fundamental Desargues Theorem. Finally, we present some interesting applications of these geometries in real life problems. This way, we are able to observe that by minimally changing the axioms, a geometry can change radically, from its most basic properties to even the curvature of a certain plane.

Keywords— geometry; axioms; Desargues Theorem; non-Desarguesian geometry; parallels; curvature; projective geometry; Euclid; planes “What has been affirmed without proof can also be denied without proof.” – Euclid

1. Introduction

When we think of the word geometry, we imagine points, lines, angles, triangles, squares or planes, all of them with a very clear and defined shape. If we are asked to draw a straight line that passes through two points, we do not stop to think about how to draw it: there is only one possible solution. If two of the angles of a triangle measure 40◦ and 90◦, and we are asked for the third angle, we answer 50◦ without thinking too much, since the angles of the triangle add up to 180◦.

But... is all this that we take for granted, true? Well, in the geometry we are teached at school, the Euclidean geometry, yes. But this is not the only existing and possible geometry. There are many perfectly consistent geometries with unique properties that, at first, we could not even imagine. For example, we could say that all straight lines intersect, parallel too, and this would be true as long as we were talking about projective geometry or elliptical geometry. Or we could ensure that the angles of a triangle always add up to less than 180◦, and we would not be saying anything strange if we were talking about hyperbolic geometry.

This report summarizes my recent school research project, which had the objective of understanding these non-Euclidean geometries, and we discuss about axiomatic geometry,

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projective geometry and non-Euclidean geometries. Also, we explain some real life applications of those geometries.

The search and analysis of consistent non-Desarguesian finite planes and consistent non-Desarguesian geometries has been the central part of this work. Desargues' theorem is one of the fundamental theorems of projective geometry, and has a trivial proof by working with the axioms in three dimensions. But it has a special feature: in two dimensions it can only be proved under some special conditions, as for example if we consider projective planes on fields. And, in fact, there are consistent planes that do not fulfill it, which are the planes that we study.

2. Axiomatic geometry

2.1 Historical introduction and Euclid's postulates. In ancient times, at the time of the Greeks, they already clearly knew how to write a proof. Based on the knowledge of geometry they already had, they elaborated new theorems and new mathematical conjectures, as we still do nowadays.

But then, Euclid, a great Greek geometer (3rd century BC), decided to compile and organize all the geometrical knowledge that had been obtained up to that point. Thus, he started from the most recently proven knowledge and theorems, and moved back to the theorems on which the others were based (since each theorem is proved from more basic ones). In this way, he arrived at some "first theorems", or some "basic knowledge" that could not be proved. These were, in other words, absolute and impossible to prove truths on which all geometry was holded, what we now call axioms.

In his famous work Elements, Euclid carried out an axiomatization of geometry: he defined five axioms, that he named postulates, as the basis of all Euclidean geometry. The five Euclid's postulates are the following (see figure 1):

I. Two different points determine a single straight line.
II. Any straight line segment can be extended indefinitely.
III. Given any point and any rectilinear segment starting from it, a circle can be drawn with center this point and radius this segment.
IV. All right angles are equal to each other.
V. If two lines intersect a third line such that the sum of the interior angles on the same side add up to less than two right angles, then the two lines intersect on the same side if extended far enough.

2.2 The fifth postulate of Euclid, Hilbert's formalization program and Gödel's incompleteness theorems. We observe that, although the first four axioms of Euclid are easy to understand quickly,

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the statement of the fifth is already a little more complicated. That is why, historically, several versions of it have been given.

But the multiple versions of this last axiom was not the only reason why this one was different from the others: many renowned mathematicians believed that this axiom could be proved from the other four, which gave rise to a very large controversy surrounding this proposition. Until the middle of the 19th century, at least 28 different proofs of Euclid's fifth postulate were published, but all of them turned out to be incorrect.

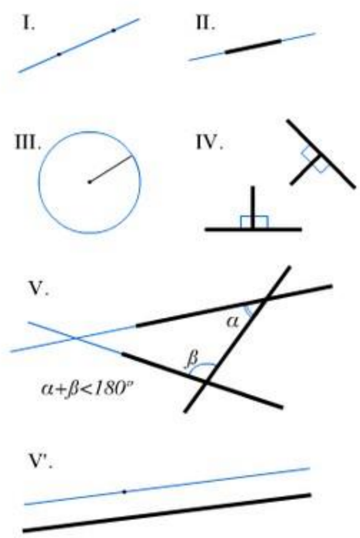

Figure 1 The five Euclid's postulates. (Source: Wikipedia)

Finally, there were mathematicians such as Lobachevsky (1792 - 1856), Bolyai (1802 - 1860), Gauss (1777 - 1855) and Riemann (1826 - 1866) who, by modifying or eliminating this last postulate, established the existence of new geometries which were perfectly consistent: the non-Euclidean geometries.

David Hilbert (1862 - 1943) was a German mathematician who initiated a formalization program, called the Hilbert Program, which was a solution to a fundamental crisis in mathematics in the 1920s. Hilbert proposed to build a whole system of axioms based on all existing mathematical theories so far, in order to form a finite and complete set of axioms on which all mathematics could be supported. In other words, he wanted to declare some basic foundations of mathematics, so that it could be fully formalized.

These foundations, or axioms, could not contradict each other, and all theorems should be able to be proved only from the axioms. In this way, one could be sure of the veracity of mathematics. Hilbert concentrated all his work in the two volumes of the book Grundlagen der Geometrie [7], where he sets out all the axioms of Euclidean geometry in two and three dimensions.

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Despite all the effort that Hilbert and the other mathematicians involved in the project made in order to axiomatize mathematics and thus remove theoretical uncertainties, it ended in failure. Gödel proved two theorems, his two incomplete theorems, that would end part of Hilbert's dreams. The theorems are the following:

1. In any consistent formalization of mathematics that is strong enough to define the concept of natural numbers, a statement can be constructed that can neither be proved nor disproved within that system.
2. No consistent system can be used to prove itself.

However, these theorems do not imply that every axiomatic system is incomplete. For example, Euclidean geometry can be fully axiomatized, since the theorem only applies to systems that allow defining the natural numbers as a set: it is not enough that the system contains them. Furthermore, the system must be able to assert that "X is a natural number" using only its axioms first-order logic. Thus, real or complex numbers also have complete axiomatizations.

These theorems had many implications in mathematics. For example, they show that if an axiomatic system can be shown to be consistent from its own axioms, then it is inconsistent. Nevertheless, Hilbert's program could be reformulated to fit these two important theorems.

2.3 Consistency of geometry. Not all geometries are possible, only those that are consistent. As explained in the book [10]:

A problem that arises whenever a mathematical theory is developed from axioms is consistency. A system of axioms is said to be consistent when from these axioms it is never possible to prove [...] two statements such that one is contradictory to the other.

However, it is clear that we cannot test all the theorems of a geometry to check if it is consistent. Thus, an arithmetic model of the geometry is constructed, and if the arithmetic is consistent, the geometry will be consistent as well. Finally, it must be shown that the arithmetic satisfies the axioms of the analyzed geometry. This arithmetic model is what we know as analytic geometry.

3. Projective geometry

3.1 Historical introduction. During the Renaissance in the 15th century, painters and artists wanted to make pictures in three dimensions, i.e., with perspective, in order to obtain more realistic works. These painters began to investigate how to project the images they wanted to represent onto a paper.

They quickly realized that very little of the actual image was preserved: neither the distances nor the angles of the drawn figures were preserved. But there was still a bigger "issue": The artists noticed that, in all the drawings, the lines seemed to converge at a point (or two, depending on the drawing) that did not exist in the actual image!

For example, if we want to draw a train track, the parallel lines end up meeting at a point (which can be inside or outside the drawing) that does not exist in reality. They named it vanishing point. Some mathematicians of the time became increasingly interested in this phenomenon and created a new geometry. They called it projective geometry, since it arose from projecting reality onto a flat surface; that is, projecting a 3D image on a 2D plane.

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3.2 Definitions and basic concepts. There are many concepts we should define and explain to understand projective geometry, but here we just state a very few basic concepts that we need for the following sections.

The first important concept of projective geometry is to note that, in Euclidean geometry, parallel lines share one thing: direction. In projective geometry, these parallels intersect at what is called the point of infinity, or the ideal point.

Definition (Cross-ratio). Let A, B, X and Y be points of a line r. The cross-ratio, or double ratio, of XY with respect to AB is denoted by (AB; XY) and is defined as

$$(AB;XY) := \frac{(AB;X)}{(AB;Y)} = \frac{\sin\alpha/\sin\beta}{\sin\alpha'/\sin\beta'},$$

where $\alpha = \widehat{AOX}$, $\beta = \widehat{XOB}$, $\alpha' = \widehat{AOY}$ *and* $\beta' = \widehat{YOB}$, *for a point of observation* O. Y *is said to be the* harmonic conjugate *of* X, *and* X *is said to be the harmonic conjugate of* Y.

From this equality we extract a very important property of the cross-ratio: it only depends on the four angles, i.e. the cross-ratio of four aligned points is a projective invariant.

Definition (Projective plane). A projective plane is a set of elements called points, together with a family of subsets called lines, that satisfy the following conditions:

- Any pair of points belong to exactly one line.
- Any pair of lines intersect in exactly one point.
- There exists a quadrilateral: a set of four points such that no subset of three of them are aligned.

3.3 Desargues' theorem. Desargues' theorem is a fundamental theorem of projective geometry. Curiously, it can only be proved in 3D. It is impossible to prove it only with the axioms of the 2D projective geometry, while its 3D proof is very simple. Therefore, the theorem must be considered one of the axioms of projective geometry in two dimensions if we work on Desarguesian planes (planes that satisfy the Desargues' theorem). The theorem states the following:

Theorem (Desargues'). In the projective plane two triangles are projective from a point if and only if they are projective from a line.

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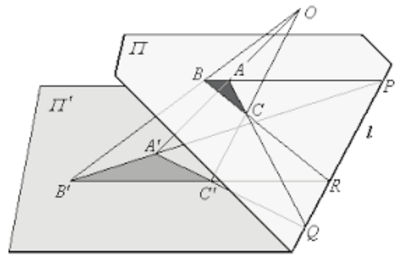


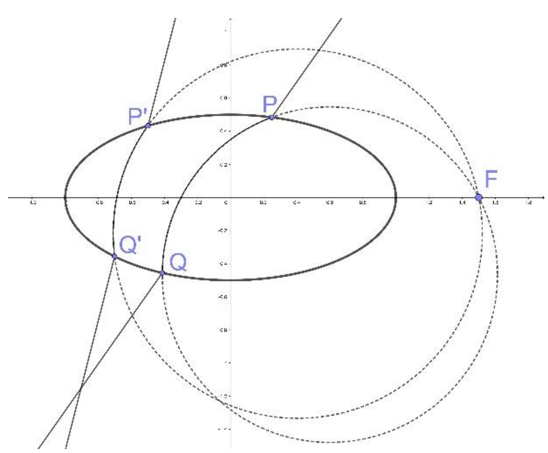
Figure 2 Representation of Desargues' theorem. (Source: Miscelánea Matemática)

This theorem holds whenever the two triangles lie in different planes. But as we see in the following sections, this is not always the case when the two triangles are coplanar.

3.4 Hilbert's non-Desarguesian geometry. Hilbert was one of the first mathematicians to try to prove the independence of Desargues' theorem with respect to the five groups of axioms. Thus, he managed to create a geometry that did not fulfill Desargues' theorem.

In this geometry, explained in [3], Hilbert considers the ellipse of semiaxes 1 and 1/2 centered at the origin, and a system of circles intersecting the x axis at the point F = (3/2,0) and only intersecting the ellipse in two points of the real plane. As points of the created geometry, consider the points of the Euclidean plane, and as lines, the lines of the Euclidean plane, except those intersecting the ellipse in two different points P and Q. On these lines, we construct the circle that passes through the points P, Q and F, and we replace the segment PQ of the line by the arc PQ of this circle. The result can be seen in figure 3(a).

To show that this is a non-Desarguesian geometry, we focus on three lines: the x axis, the y axis and the line through the origin that passes through the point x = (3/5,2/5). We then look at two triangles with the vertices on these three lines, and that are not inside the ellipse. These two triangles are projective from a line r but not from a point (the three mentioned lines do not all intersect at the same point). This can be seen in figure 3(b).



(a) Representation of non-Desarguesian Hilbert geometry

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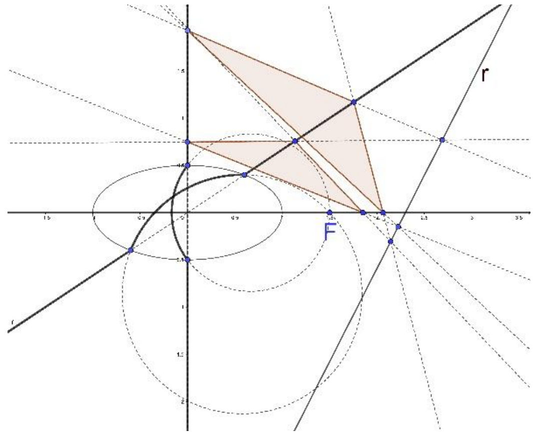

(b) This is a non-Desarguesian geometry

Figure 3 Hilbert's non-Desarguesian geometry.

3.5 Free planes. Free planes are the simplest and easiest to understand examples of non-Desarguesian planes.

As exposed in [1], we start with a configuration of points and lines that we call X_0, consisting of four points A, B, C and D, and four lines AC, CB, BD and DA. It is easy to see that X_0 is not a projective plane because it does not meet the necessary conditions exposed in its definition (see section 3.2): for example, the points A and B do not belong to the same line. Then, given a configuration X_n ($n \in N$), we construct $X_{n+1} \supset X_n$ as follows:

• The points of X_{n+1} are the points of X_n and, furthermore, for each pair of lines r and s of X_n that do not intersect, we add a new point **only** incident with the lines r and s.

• The lines of X_{n+1} are those of X_n and, furthermore, for each pair of points A and B not aligned in X_n, we add a line **only** incident with A and B.

Finally, we define the plane as:

$$X = \bigcup_{i=0}^{\infty} X_i$$

from a point (the three mentioned lines do not all intersect at the same point). This can be seen in figure 3(b).

In figure 4, we observe the construction of this plane X up to X_4.

Proposition. X is a non-Desarguesian projective plane.

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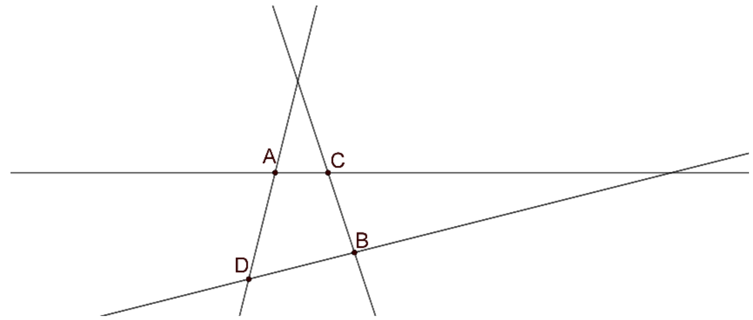
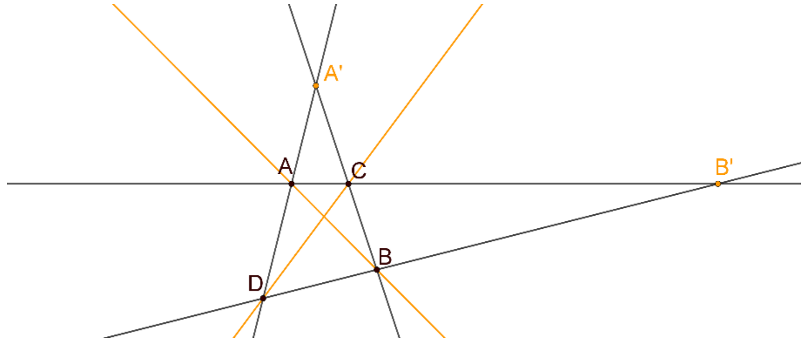
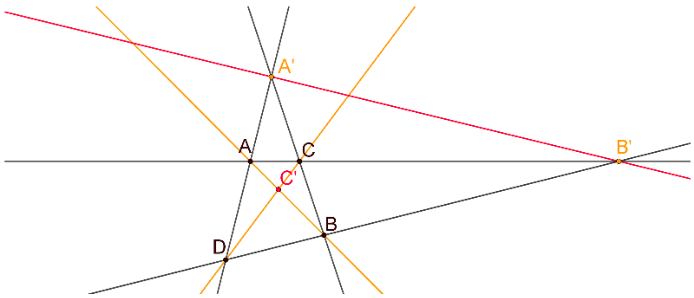


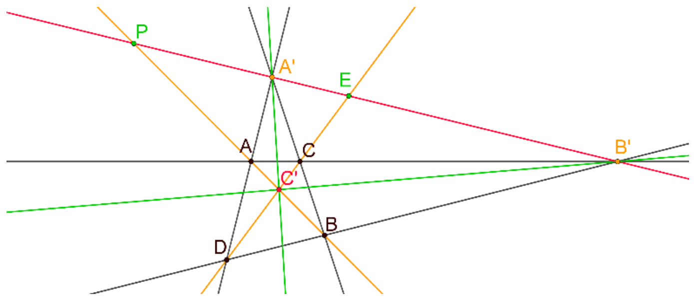
(a) ***X0***



(b) ***X1***



(c) ***X2***



(d) ***X3***

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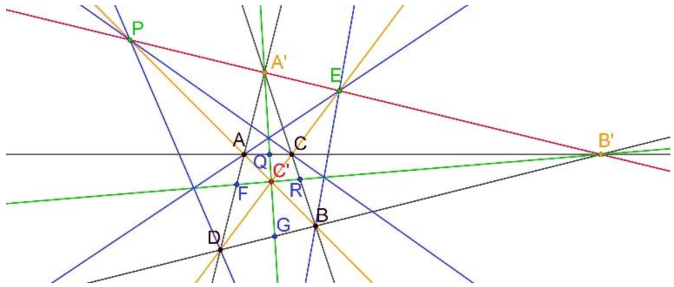

(e) ***X4***

Figure 4 Formation of a free plane up to X4

Proof. Let us look at the triangles △ ABC and △ A′, B′, C′. They are projective from the point D. In order for them to also be projective from a straight line, the points P, Q and R would have to be aligned.

But since these three points are not aligned in X_4, they will not be aligned in X either. Therefore, this plane does not satisfy Desargues' theorem.

3.6 Moulton plane. Moulton's plane was presented in 1902 by Forest Ray Moulton (1872 - 1952) in his work "A simple nonDesarguesian geometry". Its purpose was to find a simpler non Desarguesian geometry than Hilbert's.

The points that constitute the Moulton plane are the points of the real plane R 2, but with a point at the infinity for each family of parallel lines.

Lines with negative slope, horizontal or vertical, are ordinary lines, but those with slope $k > 0$ become lines with slope $k/2$ when crossing the x axis, as we see in figure 5. In this way, for example, a line in the Moulton plane through the points (0, −1) and (2,1/2) crosses the x axis at the point (1, 0).

Moulton's plane is a non-Desarguesian plane as can be clearly seen in different examples. We show two different configurations where we easily see that the theorem is not fulfilled. In the example of figure 6(a), the two triangles are projective from a point but not from a line. In the other example of figure 6(b) the two triangles are projective from a line but not from of a point.

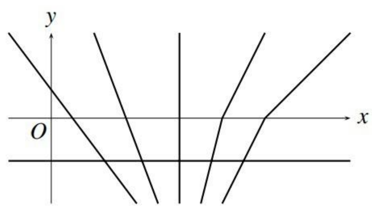

Figure 5 Lines of the Moulton plane. (Source: [11])

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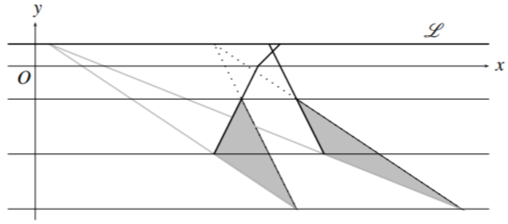
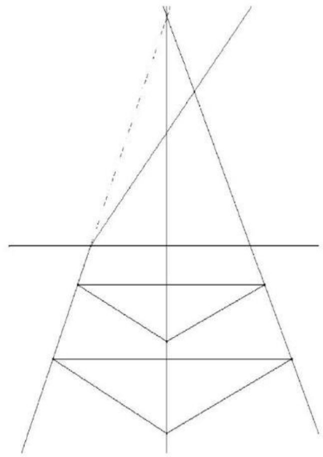


(a) Two triangles projective from the infinity point but not projective from a straight line. *(Source: [11])*



(b) Two triangles projective from the line at infinity but not from a point. *(Source: [3])*

Figure 6 The Moulton plane, a non-Desarguesian plane.

3.7 Veblen-Wedderburn plane. The Veblen-Wedderburn plane is built on the abelian group A = Z/3Z × Z/3Z consisting of nine elements: 0, 1, −1, i, −i, 1 + i, 1 − i, −1 + i, −1 − i, which multiply as shown in table 1. We can observe that the distributive property on the right is not satisfied. For example, (−1 + i)i = 1 + i ≠ −1 − i = i (−1 + i). Therefore, A is not a ring, but a near-field.

The Veblen-Wedderburn plane is the projective plane on this abelian group A: P^2 (A), which is not Desarguesian. The following proof is from [1].

Proposition. The Veblen-Weddeburn plane is a non-Desarguesian plane.

	i	$-i$	$1+i$	$1-i$	$-1+i$	$-1-i$
i	-1	1	$-1+i$	$1+i$	$-1-i$	$-1+i$
$-i$	1	-1	$1-i$	$-1-i$	$1+i$	$-1+i$
$1+i$	$1-i$	$-1+i$	-1	$-i$	i	1
$1-i$	$-1-i$	$1+i$	i	-1	1	$-i$
$-1+i$	$1+i$	$-1-i$	$-i$	1	-1	i
$-1-i$	$-1+i$	$1-i$	1	i	$-i$	-1

Table 1 Multiplication table of the elements of the abelian group forming the Veblen-Wedderburn plane.

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	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
1	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	-1	e_4	e_7	$-e_2$	e_6	$-e_5$	$-e_3$
e_2	e_2	$-e_4$	-1	e_5	e_1	$-e_3$	e_7	$-e_6$
e_3	e_3	$-e_7$	$-e_5$	-1	e_6	e_2	$-e_4$	e_1
e_4	e_4	e_2	$-e_1$	$-e_6$	-1	e_7	e_3	$-e_5$
e_5	e_5	$-e_6$	e_3	$-e_2$	$-e_7$	-1	e_1	e_4
e_6	e_6	e_5	$-e_7$	e_4	$-e_3$	$-e_1$	-1	e_2
e_7	e_7	e_3	e_6	$-e_1$	e_5	$-e_4$	$-e_2$	-1

Table 2 Multiplication of octonions.

Proof. Consider (in the affine plane A^2) two triangles △ ABC and △ A′ B′ C′, with vertices A = (0, i), B = (1, i), C = (i, 1), A′ = (−1 − i, −1 − i), B′ = (−i, −1 − i) and C′= (1 − i, i).

Then the line AA′ is y = xi + i, the line BB′ is y = −xi − i and the line CC′ is y = x (−1+i) −1+i. The three straight lines pass through the point P = (−1, 0), therefore the two triangles are perspective from this point P.

In addition, we can calculate the equations of the lines AB: y = i, and A′ B′: y = −1 − i, deducing that the two lines AB and A′ B′ are parallel. Calculating the equations of the lines AC: y = x (−1 −i) +i and A′C′: y = x (−1 − i) − i, we deduce that these two lines AC and A′C′ are also parallel. But if we calculate the equations of the lines BC: y = −x + 1 + i and B′C′: y = xi + 1 − i, we note that these two are not parallel! Consequently, the two triangles are not projective from a straight line and, therefore, the Veblen-Wedderburn plane is a non-Desarguesian plane.

3.8 Cayley plane. The Cayley plane is the projective plane P^2 (O) built on the octonions (O). In the book [1] we find several very interesting and important properties of it, but here we focus only on proving that it is a non-Desarguesian plane, and for that we proceed as in section 3.7, showing a counterexample.

First, we need to shortly introduce what the octonions are: octonions are a generalization of complex numbers in eight dimensions. They are a type of hypercomplex numbers and can be obtained from the multiplications of quaternions (O≅ H×H = H^2).

An octonion is a real linear combination of eight units $e0$, $e1$, $e2$, $e3$, $e4$, $e5$, $e6$, $e7$, where $e0$ is usually identified by the number 1. Thus, an octonion O can be written as:

$$O = x0e0 + x1e1 + x2e2 + x3e3 + x4e4 + x5e5 + x6e6 + x7e7,$$

where $x0$, ..., $x7$ are real coefficients, or as an orthogonal sum of two quaternions $h1 = a1+b1i+c1\,j+d1k$ and $h2 = a2+b2i+c2\,j+d2k$:

$$O = h1 + vh2 = a1 + b1i + c1\,j + d1k + v(a2 + b2i + c2\,j + d2k),$$

where v B (0, 1) ∈ O. We show the multiplication of octonions in table 2. The following proof is from [1].

Proposition. The Cayley plane is a non-Desarguesian plane.

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Proof. Consider (in the affine plane) two triangles $\triangle ABC$ and $\triangle A'B'C'$, with vertices $A = (0, 0)$, $B = (u, 0)$, $C = (1, u)$, $A' = (0, j)$, $B' = (u, j)$ and $C' = (1, i + j + u)$.

Then the line AA' is $x = 0$, the line BB' is $x = u$, and the line CC' is $x = 1$. Therefore, since the lines AA', BB' and CC' are parallel, the two triangles are perspective with respect to a point at infinity.

In addition, we can also calculate the equations of the lines AB: $y = 0$, $A'B'$: $y = j$. From here we deduce that the two lines are parallel. We also find the equations of the lines AC: $y = xu$, and

$A'C'$: $y = x(i + u) + j$, which intersect in the point $(-k, uk)$, where $k = i\,j$. Finally, the line BC is

$$y = \frac{x(v-1)+v+1}{2}$$

, and the line $B'C'$

$$y = \frac{xu + 2j - vu}{2}$$

, where $u = (1 + v)(i + v)$.

Therefore, the two triangles should be in perspective with respect to the line $y = vk$, which intersects the line BC at the point $(-k + v(1 - k), vk)$. But this point does not belong to the line $B'C'$. Consequently, the two triangles are not perspective from a straight line and, therefore, do not satisfy Desargues theorem.

3.9 Applications. In mathematics, projective geometry is used in many fields such as topology, group theory, etc.

However, it is also used in many other disciplines, especially in IT, computing and robotics. Next, we present some examples of work that could not be developed without projective geometry.

Computer vision and 3D reconstruction. Projective geometry has great applications in the field of computer vision and computer graphics. For example, it is used in stereovision, which is the 3D reconstruction from two 2D images.

In our work [5], we study the geometric model of a camera, and how can we explicitly model the system using projective geometry, as exposed in [6].

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The steps for 3D reconstruction are: from two images of the same object taken from two different places, we do a segmentation or extraction of all the lines and geometric elements that want to be reconstructed from the images. Then a matching between the corresponding pixels in the two images is done, using a collineation of the projective planes and the fundamental theorem of projective geometry. Then the equations of the projection of each 3D point to the two corresponding points on the two 2D images are used to write a system of equations using homogeneous coordinates or projective coordinates. This system of equations is overdetermined (4 × 3) and is solved using numerical methods, specifically using the least squares method, and thus we obtain the 3D coordinates of each point that we want to reconstruct.

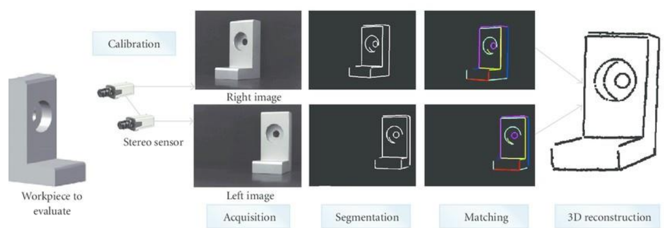

Figure 7 Steps for the 3D reconstruction 3D. (Source: [2])

3D reconstruction opens an immense range of applications, for example in the fields of medicine, architecture and virtual reality.

Measurements in photographs. In [8], the double ratio and its properties are used to measure distances based on the proportions that we can extract from photographs.

In [9], projective geometry is used to calculate the proportions of the rectangle that forms the ground floor of a building from an image of it (the image in figure 8).

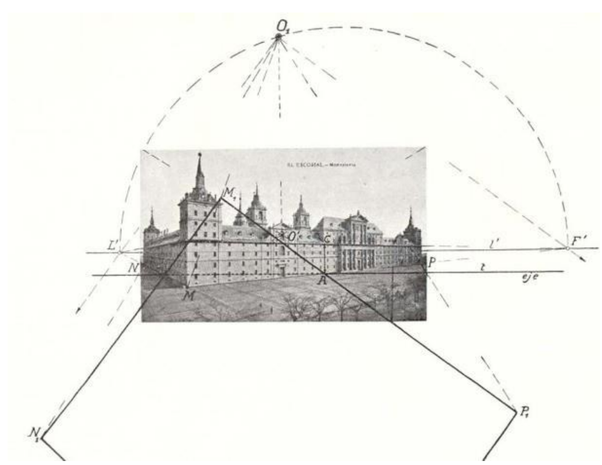

Figure 8 Calculating the proportions of the ground floor (Source: [9])

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4. Hyperbolic geometry

Hyperbolic geometry is a non-Euclidean geometry, obtained by denying Euclid's fifth postulate. Instead, we have the so-called Lobachevsky's axiom, which states the following: Axiom V' (Lobachevsky's Axiom). There exists a line r and a point P not belonging to r such that at least two lines pass through P without intersecting r.

Hyperbolic geometry arose when mathematicians tried to construct a totally consistent geometry that did not fulfill Euclid's fifth postulate, in order to prove that it was independent of the other four and that it could not be proved from those. After several attempts, finally, in the 1820s, two distinguished mathematicians, János Bolyai and Nikolai Ivanovich Lobachevski, individually succeeded in creating a completely consistent geometry denying Euclid's fifth postulate: the hyperbolic geometry.

4.1 Properties of hyperbolic geometry. Hyperbolic geometry is a geometry with negative curvature. This means that if we take any point and make two cuts perpendicular to each other in the plane through this point, the two cuts have curvatures of opposite signs, as we see in figure 9.

The most relevant characteristics of this geometry are a consequence of this negative curvature. Thus, in hyperbolic geometry, we have infinite lines parallel to a given line that have a common point, as defined by Lobachevsky's axiom. Another important feature is the fact that, in an hyperbolic plane, the angles of a triangle always add up to less than 180◦. This is represented in figure 10.

4.2 Applications. Hyperbolic planes and spaces have many applications in various fields of physics, as well as in chemistry and biology. Albert Einstein, for example, used the negative curvature of hyperbolic planes to ground his general theory of relativity. We can find hyperbolic geometry even in works of art and architecture. In addition, hyperbolic geometry is present in nature in perhaps the most unexpected way: we find hyperbolic figures both in the shape of mountains and in coral reefs, and even in lettuce leaves. It is thought that this may be due to the fact that hyperbolic planes allow more surface area in less volume, which helps the plants to get more nutrients.

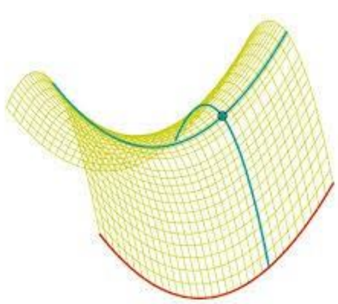

Figure 9 A point with negative curvature. (Source: Wikipedia)

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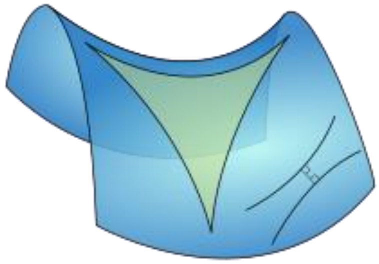

Figure 10 A triangle in hyperbolic geometry. (Source: Wikipedia)

5. Elliptic geometry

Elliptic geometry is another non-Euclidean geometry. But in this case, the axiom that replaces Euclid's fifth postulate is the following:

Axiom V'. Given a point P exterior to a line r, there is no line through P parallel to r.

5.1 Properties of elliptic geometry. Elliptic geometry is a geometry with positive curvature. This means that if we take any point and make two cuts perpendicular to each other in the plane through this point, the two cuts have curvatures of same sign.

In this case, the most relevant characteristics of this geometry are also a consequence of the positive curvature. Thus, in elliptic geometry, every pair of lines intersects at one point (if we talk about single elliptic geometry) or at two points (if we talk about double elliptic geometry as, for example, spherical geometry). Another important feature is the fact that, in an elliptic plane, the angles of a triangle always add up to more than 180◦. We represent this in figure 11.

5.2 Applications. Elliptic geometry and, in particular, spherical geometry, has a lot of applications, as we live in a spherical planet. Thus, elliptic geometry is used to analyze distances in Earth. For example, we can easily show that the shortest distance between two points in Earth is not a straight line, but a spherical line, as we observe in figure 12. In our work [5] we have a more detailed explanation of this and a computation of the shortest distance between two points of the earth, using spherical coordinates and the Haversine formula.

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Figure 11 Elliptic triangle. (Source: Wikipedia)

Figure 12 Shortest distance between BCN and NY.

6. Conclusions

In our original work [5] we present the axioms for two dimensions of the four different exposed geometries (the Euclidean, the projective, the hyperbolic and the elliptic), we compare these axioms and we draw some interesting conclusions about axiomatic geometry. Analyzing the axioms of these geometries, we observed that the fifth postulate of Euclid determines the curvature of a given plane. We were able to deduce that all these four geometries were, in fact, "cases" of projective geometry, because their axiomatic systems could be transformed into the axiomatic system of projective geometry just by adding some axioms.

In conclusion, we have understood how an axiomatic system is created, and we have exposed different geometries that are not so well known. In addition, we have presented five counterexamples to Desargues' theorem. Finally, we have also investigated some interesting applications of these geometries in several fields outside mathematics.

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What is the Pell's equation?

by Fedir Fedorov

Pell's equation is a Diophantine equation of the form $x^2 - dy^2 = 1$ where d is a given positive square-free integer. Square-free means that d is not divisible by any square number other than 1. With this setup, you try to find solutions (x, y) for this equation, where x and y are non-negative integers.

Why the restrictions with the number d? Well, if d was a square, for example $d = k^2$ for an integer k, then the equation would turn to:

$$x^2 - k^2y^2 = 1$$

$$\Leftrightarrow (x - ky)(x + ky) = 1$$

In this case, we are able to factorize the left side into the product of two integers, which is not interesting (because each integer should have the value of either 1 or -1).

Another weird restriction is the square-free part. If d was to contain another square number, so for example $d = a \cdot k^2$, where a is a square-free integer, and k is an integer bigger than 1. Then we could rewrite our Pell's equation to:

$$x^2 - ak^2y^2 = 1$$

$$\Leftrightarrow x^2 - a(ky)^2 = 1$$

As you can see, this would turn into a special case of another Pell's equation, where d (in this example a) is square-free. So it suffices to consider only square- free integers d.

Some background knowledge

Firstly, let us introduce the algebraic structure Z $[\sqrt{d}]$ which is defined as follows:

$$Z[\sqrt{d}] := \{a + b \cdot \sqrt{d} \mid a, b \in Z\}$$

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Here, we will consider these structures only for square-free integers d as well. The great thing is that (for square-free integers d), you can easily show that the described structure satisfies all the axioms of a commutative ring regarding the 'normal' addition and multiplication (this fact is left as an exercise for the reader).

Now you will probably think "Great, but what does this have to do with our Pell's equation?". And this is indeed a very good question. Considering this ring Z[$\sqrt{d}$], we can factor the Pell's equation:

$$x^2 - dy^2 = 1 \Leftrightarrow (x - y\sqrt{d})(x + y\sqrt{d}) = 1$$

Now we can see that, searching for solutions of the Pell's equation with a given d is equivalent to searching for elements of our ring Z[√d] with a multiplicative inverse (called 'units').

From one solution to infinitely many

If you look closely at Pell's equation, you might already guess one solution (x, y) = (1, 0). This is called the trivial solution and it is a solution of a Pell's equation of every kind (regardless of the value of d). But this solution is too easy and because of that, not very interesting in regard to other solutions.

Suppose that you found the minimal solution to a specific Pell's equation apart from the trivial solution (1, 0). We call this minimal pair (x_1, y_1) the fundamental solution of the equation. Now, it is possible to get infinitely many solutions to Pell's equation applying the following formula:

$$x_n + y_n\sqrt{d} = (x_1 + y_1\sqrt{d})^n$$

where x_n, y_n are also integers, and n is an arbitrary positive integer. By applying the binomial theorem, you then get:

$$x_n - y_n\sqrt{d} = (x_1 - y_1\sqrt{d})^n$$

Now, the new pair (x_n, y_n) is also a solution to Pell's equation:

$$\begin{aligned} x_n^2 - dy_n^2 &= (x_n + y_n\sqrt{d})(x_n - y_n\sqrt{d}) \\ &= (x_1 + y_1\sqrt{d})^n(x_1 - y_1\sqrt{d})^n \\ &= ((x_1 + y_1\sqrt{d})(x_1 - y_1\sqrt{d}))^n \\ &= (x_1^2 - dy_1^2)^n = 1^n = 1 \end{aligned}$$

Thus, now we have shown, that if we have one solution of Pell's equation, we can get infinitely many by applying the formula above. The formula can also be illustrated in a recursive way:

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$$\begin{aligned} x_n + y_n\sqrt{d} &= (x_1 + y_1\sqrt{d})^n \\ &= (x_1 + y_1\sqrt{d})(x_1 + y_1\sqrt{d})^{n-1} \\ &= (x_1 + y_1\sqrt{d})(x_{n-1} + y_{n-1}\sqrt{d}) \\ &= x_1x_{n-1} + x_1y_{n-1}\sqrt{d} + y_1\sqrt{d}x_{n-1} + y_1\sqrt{d}y_{n-1}\sqrt{d} \\ &= (x_1x_{n-1} + dy_1y_{n-1}) + (x_1y_{n-1} + x_{n-1}y_1)\sqrt{d} \end{aligned}$$

It follows:

$$x_n = x_1x_n{-}1 + dy_1y_n{-}1$$

$$y_n = x_1y_n{-}1 + x_n{-}1y1$$

Now we have shown that with this explicit or recursive formula, you are able to get infinitely many solutions of Pell's equation, assuming you already found one. In the following, we will prove that if (x_1, y_1) is indeed the smallest (fundamental) solution, then every other solution of Pell's equation can be illustrated with the formula derived above.

Suppose that we found another solution (x′, y′) of Pell's equation satisfying $x' + y'\sqrt{d} \neq (x_1 + y_1\sqrt{d})^n$ for every positive integer n. Then there exists a positive integer k > 0 satisfying:

$$x_k + y_k\sqrt{d} < x' + y'\sqrt{d} < x_k+1 + y_{k+1}\sqrt{d} = (x_k + y_k\sqrt{d})(x_1 + y_1\sqrt{d})$$

Now, we multiply the inequality with ($x_k - y_k$ √d), knowing that $(x_k - y_k\sqrt{d})(x_k + y_k\sqrt{d}) = x^2 - dy^2 = 1$ because (x_k, y_k) is a solution of Pell's equation:

$$\begin{aligned} 1 &< (x' + y'\sqrt{d})(x_k - y_k\sqrt{d}) < x_1 + y_1\sqrt{d} \\ \Leftrightarrow 1 &< (x'x_k - dy'y_k) + (y'x_k - x'y_k)\sqrt{d} < x_1 + y_1\sqrt{d} \end{aligned}$$

It is left for the reader to prove that, if (x′, y′) and (x_k, y_k) are solutions to Pell's equation, then ($x'x_k - dy'y_k$, $y'x_k - x'y_k$) is also a solution. Now we have found a new solution of Pell's equation, which is greater than the trivial solution (1, 0) but smaller than the fundamental solution (x1, y1). This is a contradiction to the definition of the fundamental solution.

Now we have proven that there cannot exist another solution of Pell's equation, and that every solution has the following form:

$$xn + yn\sqrt{d} = (x_1 + y_1\sqrt{d})^n$$

where n is a positive integer and (x_1, y_1) the fundamental solution.

Finding the fundamental solution

The fundamental solution can be found with the continued fraction for √d. You have to look at the sequence of convergents to the continued fraction h_i/k_i and at some point, you will reach a

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positive integer i which satisfies $h_i = x_1$, $k_i = y_1$ - the fundamental solution to Pell's equation $x^2 - dy^2 = 1$.

This follows from the fact that the fraction xn of the solutions to Pell's equation converges towards $\sqrt{d}$. This fact is also left to prove for the reader, the approach is to substitute:

$$x_n = \frac{(x_1 + y_1\sqrt{d})^n + (x_1 - y_1\sqrt{d})^n}{2}$$
$$y_n = \frac{(x_1 + y_1\sqrt{d})^n - (x_1 - y_1\sqrt{d})^n}{2\sqrt{d}}$$

Now we know a way of finding the fundamental solution, the only problem is that it may take a very long time, for example the fundamental solution for d = 61 is (x_1, y_1) = (1766319049, 226153980).

One example of a specific Pell's equation

Let us look at Pell's equation for d = 5:

$$x^2 - 5y^2 = 1$$

First, we try to find the fundamental solution by looking at the continued fraction of $\sqrt{5}$:

$$\sqrt{5} = [2; 4, 4, 4, 4, ...] = 2 + \frac{1}{4 + \frac{1}{4 + \frac{1}{4 + \dots}}}$$

Now we look at the convergences of the continued fraction. The first is 2. We substitute this into our Pell's equation to see whether this is the desired fundamental solution:

$$2^2 - 5 \cdot 1^2 = 4 - 5 = -1$$

This is not our desired solution, so we look at the second fraction, which is 9:

$$9^2 - 5 \cdot 4^2 = 81 - 80 = 1$$

The result is 1, so we have found our fundamental solution (x1, y1) = (9, 4). The other solutions can be derived using for example the recursive formula for the solutions of Pell's equation:

$$x_2 = x_1x_1 + dy_1y_1 = 9 \cdot 9 + 5 \cdot 4 \cdot 4 = 81 + 80 = 161$$

$$y_2 = x_1y_1 + x_1y_1 = 9 \cdot 4 + 9 \cdot 4 = 36 + 36 = 72$$

And we see that (161, 72) is indeed a solution of Pell's equation for d = 5: 1612 − 5 · 722 = 25921 − 25920 = 1

With this recursive formula, we are able to find infinitely many solutions to this specific example of Pell's equation.

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Conclusion

I hope that this article was in any way helpful for you to get an understanding of Pell's equation.

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